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Nonlinear internal gravity waves in a slowly varying medium

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Nonlinear internal gravity waves in an inviscid incompressible fluid are discussed for the case when the properties of the medium vary slowly on a scale determined by the local wave structure. A two-time-scale technique is used to obtain transport equations which describe the slowly varying modulations of the waves. Various solutions of these transport equations are discussed.

1. Introduction

Internal gravity waves are an important feature of the dynamics of the ocean and of the atmosphere; Phillips (1969, §5.1) has documented observations made in the ocean, while Bretherton (1966) has done likewise for the atmosphere. For an inviscid incompressible fluid with constant Brunt–Väisälä frequency the linearized equations of motion have plane wave solutions whose properties are well known (Phillips 1969, §5.4; Bretherton 1969). When the Brunt–Väisälä frequency is not constant but is slowly varying with respect to the wavelength, and also when there is a mean flow, similarly slowly varying, the waves may be regarded as having a local structure determined by the plane wave solution while the wave parameters vary slowly on a scale determined by the non-uniform background. For linearized internal gravity waves this approach has been used by Bretherton (1966) and Garrett (1968).

In this paper we shall consider nonlinear internal gravity waves. As the fluid is incompressible these are transverse, sinusoidal and identical to the linearized solutions, with the important exception that there is now a wave-induced mean flow proportional to the square of the wave amplitude. The purpose of this paper is to study slowly varying modulations of these waves; to this end we use the familiar two-time-scale technique (Whitham 1970). The local structure of the wave is assumed to be described by the nonlinear plane wave solution while the wave frequency, wavenumber, amplitude and the mean flow vary on the time and length scales associated with the equilibrium density stratification. This approach has also been used by Drazin (1969) and Rarity (1969); however, both these authors omitted the nonlinear contribution to the mean flow. Bretherton (1969) also considered nonlinear waves; he used a perturbation expansion in the wave amplitude and obtained results correct to the second order in the wave amplitude. To this order his results agree with ours. To find the transport equations which govern the modulations we use an averaged variational principle (see Whitham 1970). It will be shown that these equations consist of the local dispersion relation, corrected for the Doppler effect due to the mean flow, an equation for conservation of wave action, and a set of equations which govern the mean vorticity production associated with the mean flow.

In § 2 the equations of motion are formulated as the Euler equations of a certain variational principle, using the Clebsch transformation. In § 3 the non-linear plane wave solution (for constant Brunt–Väisälä frequency) is derived. In § 4 the transport equations which govern slowly varying modulations are obtained from an averaged variational principle. In § 5 various solutions of these transport equations are discussed. In §§ 5.1 and 5.2 modulations which depend on the vertical co-ordinate and the time alone are considered and the results are used to discuss the mechanism of critical-layer absorption. These results are identical to those obtained in the linearized theory, except when it is also assumed that there are no horizontal pressure gradients. In § 5.3 two-dimensional modulations are briefly discussed. In § 5.4 small amplitude three-dimensional modulations are discussed and it is shown that the wave is unstable to certain horizontal modulations, perpendicular to the wavenumber vector.

2. The equations of motion and the Clebsch transformation

Assuming that the fluid is incompressible and inviscid, the equations of motion are

$$\nabla . \mathbf{v} = 0, \tag{2.1}$$

$$d\rho/dt = 0, \tag{2.2}$$

$$\rho\, d\mathbf{v}/dt + \nabla p + \rho \nabla \chi = 0, \tag{2.3}$$

where $\mathbf{v}$ is the velocity of the fluid particle which is at $\mathbf{x}$ at time t, ρ is its density, p is the pressure and χ is the gravitational potential gz. The first task is to recast these equations using the Clebsch transformation (see Lamb (1932, § 167), where the barotropic case is discussed). Let the fluid particle which will be at the level z at time t be at the level z_0 at the initial time $t = 0$. In the initial state let

$$\rho = \rho_0(z_0). \tag{2.4}$$

Then (2.2) implies that (2.4) holds for all subsequent times t.

If $p_0(z_0)$ is defined so that

$$dp_0/dz_0 = -g\rho_0(z_0) \tag{2.5}$$

the vorticity equation is

$$d\mathbf{w}/dt = \mathbf{w} . \nabla \mathbf{u} + \nabla(p - p_0) \times \nabla(1/\rho_0), \tag{2.6}$$

where

$$\mathbf{w} = \nabla \times \mathbf{v}. \tag{2.7}$$

Let the initial vorticity $\mathbf{w}_0$ be given by the Clebsch representation†

$$\mathbf{w}_0 = \nabla\gamma \times \nabla\delta. \tag{2.8}$$

† Equation (2.8) is, in general, valid only locally; if the vortex lines associated with $\mathbf{w}_0$ are sufficiently knotted, then further potentials, analogous to γ and δ must be added to (2.8), cf. Bretherton (1970) and Moffatt (1969).

Then (2.6) may be integrated with respect to time along the particle paths to give

$$\mathbf{w} = \nabla\gamma \times \nabla\delta + \nabla\left(\int_0^t (p-p_0)\,dt\right) \times \nabla(1/\rho_0), \tag{2.9}$$

and so

$$\mathbf{v} = \nabla\overline{\phi} + \gamma\nabla\delta + \left(\int_0^t (p-p_0)\,dt\right)\nabla(1/\rho_0). \tag{2.10}$$

On introducing the change of variables

$$\phi = \overline{\phi} - \beta\alpha, \tag{2.11}$$

$$\beta = -\left(\int_0^t (p-p_0)\,dt\right)\frac{d}{dz_0}(1/\rho_0), \tag{2.12}$$

$$\alpha = z_0 - z, \tag{2.13}$$

we find that

$$\mathbf{v} = \nabla\phi + \alpha\nabla\beta - \beta\mathbf{k} + \gamma\nabla\delta, \tag{2.14}$$

where $\mathbf{k}$ is a unit vector in the z direction. Substitution into (2.3) then gives

$$(p-p_0)/\rho_0 = g\alpha - \{\phi_t + \alpha\beta_t + \gamma\delta_t + \tfrac{1}{2}|\mathbf{v}|^2\}. \tag{2.15}$$

Also, (2.12) and (2.13) may be replaced by

$$\frac{d\beta}{dt} = -(p-p_0)\frac{d}{dz_0}(1/\rho_0), \tag{2.16}$$

$$d\alpha/dt = -w, \tag{2.17}$$

where $w = \mathbf{v}.\mathbf{k}$ is the z component of velocity, while γ and δ, being constant along particle paths, are given by

$$d\gamma/dt = 0, \tag{2.18}$$

$$d\delta/dt = 0. \tag{2.19}$$

The equations of motion are now (2.1) and (2.14)–(2.19), z_0 being given by (2.13). Although these equations have the disadvantage that the representation (2.14) for the velocity is nonlinear, they have the compensating advantages that the pressure perturbation is given explicitly by (2.15) and that the vorticity is described by the potentials γ and δ, representing the initial vorticity, and by the potentials α and β, representing the production of vorticity due to the density stratification. It may be noted that α is the vertical displacement of a fluid particle. Also, in the present context, these equations have the advantage that they are the Euler equations of the variational principle (see Seliger & Whitham 1968)

$$\delta\int p\,d\mathbf{x}\,dt = 0, \tag{2.20}$$

where it is understood that p is given by (2.16) and $\mathbf{v}$ by (2.14). Indeed the variation of ϕ gives (2.1), the variation of α and of β give (2.16) and (2.17) respectively, and the variation of γ and of δ give (2.19) and (2.18) respectively.

The arguments of subsequent sections will be clarified by the introduction of a small parameter ϵ characterizing the slow variation of the density stratification.

Thus we introduce a length scale L characterizing a typical wavelength and a time scale N_0^{-1} defined by

$$N_0^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz_0}\bigg|_{z_0=0}. \tag{2.21}$$

N_0 is the Brunt–Väisälä frequency at the level $z_0 = 0$. Dimensionless variables based on the length scale L, time scale N_0^{-1}, a velocity scale N_0L and a pressure scale gL are now introduced. In addition we put

$$\rho_0 = \rho_0(\epsilon z_0), \quad p_0 = p_0(\epsilon z_0) \tag{2.22}$$

so that

$$\epsilon p_0' = -\rho_0 \tag{2.23}$$

and

$$\rho_0' = -N^2\rho_0, \tag{2.24}$$

where z_0 is now a dimensionless variable, a prime denotes differentiation with respect to ϵz_0, and

$$\epsilon = N_0^2L/g \tag{2.25}$$

is a small parameter. Thus p_0, ρ_0 and N^2 are all functions of ϵz_0, i.e. of $\epsilon(z+\alpha)$, and are slowly varying.

The equations of motion become

$$\mathbf{v} = \nabla\phi + \alpha\nabla\beta - \beta\mathbf{k} + \gamma\nabla\delta, \tag{2.26}$$

$$d\alpha/dt = -w, \tag{2.27}$$

$$d\beta/dt = -PN^2, \tag{2.28}$$

$$d\gamma/dt = d\delta/dt = 0, \tag{2.29}$$

$$(p-p_0)/\rho_0 = P = \alpha - \epsilon Q, \quad Q = \phi_t + \alpha\beta_t + \gamma\delta_t + \tfrac{1}{2}|\mathbf{v}|^2, \tag{2.30}$$

together with (2.1). In the limit $\epsilon \to 0$, we see that in (2.28) P is replaced by α and N^2 is replaced by a constant. We shall refer to this limit as the Boussinesq approximation since it involves the neglect of the inertial contribution to the pressure compared with the buoyancy contribution. However, it may be noted that the Boussinesq approximation is also widely used in situations where the density stratification is not slowly varying. In §§4 and 5 we shall consider mean flows which also vary slowly on a length scale ϵ^{-1}; the Richardson number, being the ratio of N^2 to the square of the rate of shear, is then of $O(\epsilon^{-2})$.

3. Plane waves

Allowing $\epsilon \to 0$, we seek a plane wave solution to the equations of motion. Thus we put

$$\mathbf{v} = \mathbf{V} + \hat{\mathbf{v}}(\theta), \tag{3.1}$$

$$\alpha = \hat{\alpha}(\theta), \tag{3.2}$$

$$\beta = B + \hat{\beta}(\theta), \tag{3.3}$$

where

$$\theta = \boldsymbol{\kappa}.\mathbf{x} - \omega t. \tag{3.4}$$

$\hat{\mathbf{v}}$, $\hat{\alpha}$ and $\hat{\beta}$ are periodic functions of the phase θ of period 2π and have zero mean, so that $\langle\mathbf{v}\rangle = \mathbf{V}$ and $\langle\beta\rangle = B$ are constants, where $\langle\,\rangle$ denotes an average with respect to θ over 2π; it may be shown that $\langle\alpha\rangle = 0$. ω is the frequency and $\boldsymbol{\kappa}$, with components (l, m, n), is the wavenumber vector. Without loss of generality we may put $\gamma = \delta = 0$ in this section. The solution for ϕ consistent with (3.1) is

$$\phi = \mathbf{\Pi}.\mathbf{x} - \Sigma t + \hat{\phi}(\theta), \tag{3.5}$$

so that

$$\hat{\mathbf{v}} = \boldsymbol{\kappa}(\hat{\phi}_\theta + \hat{\alpha}\hat{\beta}_\theta - \langle\hat{\alpha}\hat{\beta}_\theta\rangle) - \hat{\beta}\mathbf{k}, \tag{3.6}$$

and

$$\mathbf{V} = \mathbf{\Pi} + \boldsymbol{\kappa}\langle\hat{\alpha}\hat{\beta}_\theta\rangle - B\mathbf{k}. \tag{3.7}$$

Equations (3.6) and (3.7) may be regarded as determining $\hat{\phi}$ and $\mathbf{\Pi}$ respectively.

Substitution of (3.1) into (2.1) yields

$$\hat{\mathbf{v}}.\boldsymbol{\kappa} = 0, \tag{3.8}$$

so that

$$\hat{\mathbf{v}} = \hat{\beta}(n\boldsymbol{\kappa}/\kappa^2 - \mathbf{k}), \tag{3.9}$$

where $\kappa = |\boldsymbol{\kappa}|$. Equation (3.8) implies that

$$d/dt = -\omega'\,\partial/\partial\theta, \tag{3.10}$$

where

$$\omega' = \omega - \boldsymbol{\kappa}.\mathbf{V} \tag{3.11}$$

is the intrinsic frequency, i.e. the frequency in a medium at rest relative to the wave. Thus (3.8) ensures that the nonlinear terms in the equations of motion (for $\epsilon \to 0$) are identically zero and the plane wave solution is exactly that obtained from the linearized equations, with the exception that $\mathbf{V}$ contains the nonlinear term $\langle\hat{\alpha}\hat{\beta}_\theta\rangle$. Equation (2.27), when averaged with respect to θ, shows that

$$\mathbf{V}.\mathbf{k} = 0, \tag{3.12}$$

and so the mean velocity $\mathbf{V}$ is horizontal. Equation (3.12) may be regarded as determining B. Also, (2.28), when averaged with respect to θ, confirms that $\langle\alpha\rangle = 0$. Finally (2.27) and (2.28) imply that

$$\hat{\alpha} = a\sin\theta, \tag{3.13}$$

$$\hat{\beta} = -(aN^2/\omega')\cos\theta \tag{3.14}$$

and

$$\omega'^2 = N^2(1 - n^2/\kappa^2). \tag{3.15}$$

a is the amplitude and (3.15) is the familiar dispersion relation for internal gravity waves. These results agree with those of Rarity (1969) and Drazin (1969), except that both these authors neglected the mean velocity $\mathbf{V}$. For a plane wave this is legitimate as a *constant* mean velocity can be removed by a Galilean transformation; however it is essential to include $\mathbf{V}$ when considering modulated waves. Here we have

$$\mathbf{V} = \mathbf{\Pi}_H + \mathbf{U}, \tag{3.16}$$

where

$$\mathbf{U} = \boldsymbol{\kappa}_H \mathscr{F}/\rho_0 \tag{3.17}$$

and

$$\mathscr{F} = \mathscr{E}/\omega', \quad \mathscr{E} = \tfrac{1}{2}\rho_0 N^2 a^2. \tag{3.18}$$

A subscript H denotes the horizontal component. $\mathscr{F}$ is the wave action density (Bretherton & Garrett 1969; Garrett 1968) and $\mathscr{E}$ is the wave energy density.†

† The energy density is $\epsilon(\frac{1}{2}\rho_0|\mathbf{v}|^2 + \Upsilon)$, where $\epsilon\Upsilon = p_0(\epsilon z) - p_0(\epsilon z_0) - \alpha\rho_0(\epsilon z_0)$ is the potential energy/unit volume due to the displacement of a fluid particle from the level z_0 to z; $\Upsilon = \frac{1}{2}\rho_0 N^2\alpha^2 + O(\epsilon)$ and the wave average of the energy density is $\epsilon(\mathscr{E} + O(\epsilon^2))$.

The group velocity $\mathbf{c}$ is defined as $\nabla_\kappa \omega'$ and is given by

$$\mathbf{c} = \frac{N^2 n}{\omega' \kappa^2}\left(\frac{n\boldsymbol{\kappa}}{\kappa^2} - \mathbf{k}\right). \tag{3.19}$$

Thus the group velocity is in the plane of $\mathbf{k}$ and $\boldsymbol{\kappa}$ and is perpendicular to $\boldsymbol{\kappa}$. As $\omega' \to 0$, the wavenumber vector approaches the vertical and $\mathbf{c}$ becomes finite and horizontal. As $\omega' \to N$ the wavenumber vector approaches the horizontal and $\mathbf{c}$ tends to zero.

4. Modulated waves

If ϵ is small, but non-zero, then plane wave solutions are no longer possible. However we may consider an asymptotic solution, which is locally a plane wave, whose properties change on a length scale of $O(\epsilon^{-1})$. Thus we let

$$\mathbf{X} = \epsilon \mathbf{x}, \quad T = \epsilon t \tag{4.1}$$

and seek a solution of the form

$$\mathbf{v} = \mathbf{V}(\mathbf{X}, T, \epsilon) + \hat{\mathbf{v}}(\mathbf{X}, T; \theta; \epsilon), \tag{4.2}$$

where the phase θ is defined so that the local frequency $\omega = -\theta_t$ and the local wavenumber $\boldsymbol{\kappa} = \nabla_{\mathbf{x}}\theta$ are functions of $\mathbf{X}$ and T, and so are slowly varying. Thus

$$\theta = (1/\epsilon)\,\Theta(\mathbf{X}, T; \epsilon) \tag{4.3}$$

and

$$\omega = -\Theta_T, \quad \boldsymbol{\kappa} = \nabla\Theta. \tag{4.4}$$

(In this and the next section all spatial and time derivatives are with respect to $\mathbf{X}$ and T.) As in § 2, $\hat{\mathbf{v}}$ is to be periodic in θ with period 2π and has zero mean. For the remaining variables we put

$$\alpha = \epsilon A(\mathbf{X}, T) + a(\mathbf{X}, T)\sin\theta + \ldots, \tag{4.5}$$

$$\beta = B(\mathbf{X}, T) + b(\mathbf{X}, T)\cos\theta + \ldots, \tag{4.6}$$

$$\gamma = C(\mathbf{X}, T) + \epsilon c(\mathbf{X}, T)\sin\theta + \ldots, \tag{4.7}$$

$$\delta = (1/\epsilon)\,D(\mathbf{X}, T) + d(\mathbf{X}, T)\sin\theta + \ldots, \tag{4.8}$$

$$\phi = (1/\epsilon)\,\Phi(\mathbf{X}, T) + (nb/\kappa^2)\sin\theta - \tfrac{1}{4}ab\sin 2\theta - Cd\sin\theta + \ldots, \tag{4.9}$$

where the omitted terms are of a higher order with respect to ϵ than the displayed terms. The terms in ϕ have been chosen to ensure that $\hat{\mathbf{v}}.\boldsymbol{\kappa}$ is of $O(\epsilon)$, in agreement with (3.8); it is not necessary to do this at this stage but use of (4.9) greatly simplifies the subsequent analysis. Clearly the θ dependence of the solution, to the lowest order in ϵ, will be exactly that described by the plane wave solution of the previous section. This fact has enabled us to omit various $\cos\theta$ and $\sin\theta$ terms from the expressions above. On substituting these expressions into (2.26) we find that

$$\mathbf{V} = \nabla\phi + C\nabla D - B\mathbf{k} - \tfrac{1}{2}ab\boldsymbol{\kappa} + O(\epsilon^2), \tag{4.10}$$

$$\hat{\mathbf{v}} = b\cos\theta(n\boldsymbol{\kappa}/\kappa^2 - \mathbf{k}) + O(\epsilon). \tag{4.11}$$

The solutions for γ and δ, which could be ingored for the plane wave solution but must be included here, are most readily found by substituting (4.7) and (4.8) into (2.29). Averaging with respect to θ gives

$$\left.\begin{aligned} C_T+\mathbf{V}.\nabla C &= O(\epsilon^2), \\ D_T+\mathbf{V}.\nabla D &= O(\epsilon^2), \end{aligned}\right\} \tag{4.12}$$

while c and d are given by

$$\left.\begin{aligned} \omega' c &= b(n\boldsymbol{\kappa}/\kappa^2-\mathbf{k}).\nabla C+O(\epsilon), \\ \omega' d &= b(n\boldsymbol{\kappa}/\kappa^2-\mathbf{k}).\nabla D+O(\epsilon). \end{aligned}\right\} \tag{4.13}$$

These results confirm the form chosen for γ and δ in (4.7) and (4.8). Equation (4.12) shows that C and D are transported with the mean velocity $\mathbf{V}$, while c and d are determined by the modulations of C and D transverse to $\boldsymbol{\kappa}$.

The global behaviour (i.e. $\mathbf{X}$, T dependence) of the remaining variables can also be determined by substitution into the exact equations. However, a simpler procedure is to use the averaged variational principle. It has been shown by Bisshopp (1969) and Whitham (1970) that if the exact equations of motion are the Euler equations of a Lagrangian, in this case p (see equation (2.20)), then the global behaviour of a modulated wave is determined from the averaged variational principle

$$\delta\int\langle p\rangle\, d\mathbf{X}\, dT = 0, \tag{4.14}$$

where
$$\langle p\rangle = \frac{1}{2\pi}\int_0^{2\pi} p\, d\theta. \tag{4.15}$$

The variations in (4.14) are with respect to Θ and a, b etc. Remarkably, (4.14) is valid to all orders in ϵ. Thus

$$\langle p\rangle = \langle p_0(Z+\epsilon\alpha)\rangle+\langle\alpha\rho_0(Z+\epsilon\alpha)\rangle-\langle\epsilon\rho_0(Z+\epsilon\alpha)\,Q\rangle \tag{4.16}$$

and
$$\langle Q\rangle = \Phi_T+CD_T+\tfrac{1}{2}|\mathbf{V}|^2+\tfrac{1}{2}\omega ab+\tfrac{1}{4}b^2(1-n^2/\kappa^2)+O(\epsilon^2). \tag{4.17}$$

The global behaviour of the modulated wave is now obtained from the variation of (4.16) with respect to a, b, A, B, C, D, Φ and Θ. Thus we have

$$\omega' b = -N^2 a+O(\epsilon^2) \quad \text{from} \quad \delta a, \tag{4.18}$$

$$\omega' a = -(1-n^2/\kappa^2)\,b+O(\epsilon^2) \quad \text{from} \quad \delta b, \tag{4.19}$$

$$\mathbf{V}.\mathbf{k} = O(\epsilon^2) \quad \text{from} \quad \delta B, \tag{4.20}$$

$$\nabla.\mathbf{V} = 0 \quad \text{from} \quad \delta\Phi, \tag{4.21}$$

$$\mathscr{F}_T+\nabla.[\mathscr{F}(\mathbf{c}+\mathbf{V})] = O(\epsilon^2) \quad \text{from} \quad \delta\Theta, \tag{4.22}$$

where $\mathscr{F}$ is defined by (3.18). The variations δC and δD reproduce (4.13). The variation δA involves higher order terms in Q than those displayed; it may be shown that

$$N^2 A = N^2\langle Q\rangle-\tfrac{1}{2}(N^2)'\,a^2-\{B_T+\mathbf{V}.\nabla B+\nabla.[\tfrac{1}{2}b^2(n\boldsymbol{\kappa}/\kappa^2-\mathbf{k})]\}+O(\epsilon). \tag{4.23}$$

In all these expressions $N^2 = N^2(Z)$. Equations (4.18), (4.19) and (4.20) reproduce the local plane wave solution obtained in § 3, while (4.21), (4.22) and (4.13) describe the global behaviour of the modulations. The equation for A uncouples from the other equations and need not concern us further. The set of equations which describe the global behaviour will be called the transport equations and are discussed in the next section.

5. Transport equations

The transport equations derived in §4 are, with some simplifications and the omission of the error terms of $O(\epsilon^2)$,

$$\omega'^2 = N^2(1-n^2/\kappa^2), \tag{5.1}$$

$$\omega' = \omega - \boldsymbol{\kappa}_H.\mathbf{V}, \tag{5.2}$$

$$\mathscr{F}_T + \nabla.[\mathscr{F}(\mathbf{c}+\mathbf{V})] = 0, \tag{5.3}$$

$$\nabla_H.\mathbf{V} = 0, \tag{5.4}$$

$$\mathbf{V} = \nabla_H\Phi + C\nabla_H D + \mathbf{U}, \quad \mathbf{U} = \mathscr{F}/\rho_0\boldsymbol{\kappa}_H, \tag{5.5}$$

$$\mathscr{D}C/\mathscr{D}T = \mathscr{D}D/\mathscr{D}T = 0, \tag{5.6}$$

where

$$\mathscr{D}/\mathscr{D}T \equiv \partial/\partial T + \mathbf{V}.\nabla_H. \tag{5.7}$$

In these equations a subscript H denotes the horizontal component, so that, for example, $\nabla_H \equiv (\partial/\partial X, \partial/\partial Y, 0)$ and $\boldsymbol{\kappa}_H \equiv (l, m, 0)$. Equation (5.3) is the equation for conservation of wave action $\mathscr{F}$ and has been obtained previously for linearized internal gravity waves by Bretherton (1966) and Garrett (1968); equations like (5.3) have been discussed in a more general context by Bretherton & Garrett (1969).

The mean velocity $\mathbf{V}$ is horizontal and non-divergent. It contains a wave-induced velocity $\mathbf{U}$, which is a nonlinear term, being proportional to a^2. The other terms in (5.5) describe the transport of mean vorticity with the mean velocity. This is most readily seen by introducing the mean circulation

$$\mathscr{C} = \int_\Gamma \mathbf{V}.d\mathbf{X}, \tag{5.8}$$

where Γ is a horizontal circuit moving with velocity $\mathbf{V}$. Then (5.5) implies that

$$\mathscr{C} = \mathscr{C}_0 + \int_\Gamma \mathbf{U}.d\mathbf{X}, \tag{5.9}$$

where

$$\mathscr{C}_0 = \int_\Gamma C\nabla_H D.d\mathbf{X}. \tag{5.10}$$

$\mathscr{C}_0$ is independent of time by (5.6), and is thus equal to the initial circulation (i.e. that due to the mean vorticity before the arrival of the waves). Equation (5.10) was obtained by Bretherton (1969), who considered small amplitude internal gravity waves. The present result contains no restriction on the amplitude a other than that we must be able to neglect the error terms $O(\epsilon^2a^2)$ in comparison with the terms retained, which are $O(a^2)$ and $O(a^4)$; we require, therefore, that

$\epsilon^2 \ll a^2$, which is certainly satisfied if a is $O(1)$, for example. Equation (5.5) also implies that

$$\rho_0 \mathscr{D}\mathbf{V}/\mathscr{D}T + \nabla.(\mathscr{F}\mathbf{c}\boldsymbol{\kappa}_H) = -\nabla_H \mathscr{P}, \tag{5.11}$$

where $$\mathscr{P} = -(\rho_0\langle Q\rangle + \tfrac{1}{4}\rho_0 N^2 a^2) = -\rho_0(\mathscr{D}\Phi/\mathscr{D}T - \tfrac{1}{2}|\mathbf{V}|^2) \tag{5.12}$$

is the averaged pressure perturbation, i.e. $\epsilon^{-1}[p - p_0(Z)]$. Also

$$\mathscr{F}\mathbf{c}\boldsymbol{\kappa}_H = \langle \rho_0 \mathbf{v}\mathbf{v}_H\rangle \tag{5.13}$$

and is that component of the Reynolds stress which can exert a force in a horizontal direction. Equation (5.12) can also be obtained by averaging the momentum equation (2.3), and was obtained using this approach by Bretherton (1969) for small amplitude internal gravity waves. Bretherton (1969) gives a comprehensive discussion of the physical interpretation of (5.9) and (5.11). Here we merely comment that the principal effect of the induced velocity $\mathbf{U}$ is the production of mean vorticity in the vicinity of a wave packet, a phenomenon succinctly described by (5.9).

Φ may be eliminated from (5.6) to give

$$\mathbf{k}.\nabla\times\mathbf{V} = \mathbf{k}.\nabla\times\mathbf{U} + \mathbf{k}.\nabla_H C\times\nabla_H D. \tag{5.14}$$

A further simplification is to introduce a stream function Ψ for $\mathbf{V}$ such that

$$\mathbf{V} = \nabla\times(\Psi\mathbf{k}). \tag{5.15}$$

Equation (5.4) is now automatically satisfied and (5.14) becomes

$$\nabla_H^2\Psi + (\nabla_H\mathscr{F}/\rho_0).\boldsymbol{\kappa}_H\times\mathbf{k} + \mathbf{k}.\nabla_H C\times\nabla_H D = 0. \tag{5.16}$$

Equation (5.16) shows that in general (for exceptions see §§ 5.2.2 and 5.3) the induced velocity affects the mean flow only when the wave packet is modulated in a horizontal direction which is perpendicular to $\boldsymbol{\kappa}_H$.

The dispersion equation (5.1) is to be regarded as a partial differential equation for the phase Θ, since $\omega = -\Theta_T$ and $\boldsymbol{\kappa} = \nabla\Theta$. However, it is often convenient to regard ω and $\boldsymbol{\kappa}$ as the primary dependent variables, in which case (5.1) is supplemented by the compatibility equations

$$\boldsymbol{\kappa}_T + \nabla\omega = 0, \quad \nabla\times\boldsymbol{\kappa} = 0. \tag{5.17}$$

In the remainder of this section the transport equations are analysed under various further hypotheses.

5.1. *Quasi-steady modulations*

This is the situation which arises when ω, $\boldsymbol{\kappa}$, a and $\mathbf{V}$ are assumed to be functions of Z only. Equation (5.17) implies that ω and $\boldsymbol{\kappa}_H$ are constants. Clearly we may allow $\mathbf{V} = \mathbf{V}(Z)$ to be an arbitrary shear flow. Thus ω' can be regarded as a prescribed function of Z, and (5.1) determines n as a function of Z. The solution of (5.3) is

$$W\mathscr{F} = \text{constant}, \quad W = \mathbf{c}.\mathbf{k}, \tag{5.18}$$

and from (3.18) it follows that

$$\rho_0 a^2\omega'(N^2 - \omega'^2)^{\frac{1}{2}} = \text{constant}. \tag{5.19}$$

The solution fails where $\omega' = 0$ and also where $|\omega'| = N$; Bretherton (1966) has discussed the nature of this breakdown in detail. Even when $\mathbf{V}$ and N are constant (5.19) shows that the amplitude a grows with height as $\rho_0^{-\frac{1}{2}}$ (cf. Drazin (1969), who obtained (5.20) for the case $\mathbf{V} = \mathbf{0}$).

5.2. (Z, T)-*dependent modulations*

5.2.1. *Modulations supported by horizontal pressure gradients.* We now suppose that ω, $\boldsymbol{\kappa}$, a and $\mathbf{V}$ are functions of Z and T alone. Equations (5.4) and (5.5) impose no apparent restriction on $\mathbf{V} = \mathbf{V}(Z, T)$, which we therefore assume to be an arbitrary time-dependent shear flow. However (5.11) then implies that $\nabla_H \mathscr{P}$ is non-zero and the flow is supported by horizontal pressure gradients (this comment applies to § 5.1 as well); the case when there are no horizontal pressure gradients is examined in § 5.2.2. In the present case there are no nonlinear effects and the theory is identical to that for linearized internal gravity waves, described by Bretherton (1966).

Equation (5.17) implies that $\boldsymbol{\kappa}_H$ is a constant and that

$$n_T + \omega_Z = 0. \tag{5.20}$$

From (5.2) we have

$$n_T + W n_Z + \frac{\partial \omega'}{\partial Z} + \boldsymbol{\kappa}_H \cdot \frac{\partial \mathbf{V}}{\partial Z} = 0, \tag{5.21}$$

where $W = \mathbf{c}.\mathbf{k}$ is the vertical component of the group velocity and is a known function of n and Z, and $\partial\omega'/\partial Z$ is the explicit derivative of ω' with respect to Z through the dependence of ω' on $N^2(Z)$. Equation (5.21) is a single equation for n and is solved subject to the initial condition that at $T = 0$, $n = n_0(Z)$. Equation (5.3) then becomes

$$\mathscr{F}_T + (W\mathscr{F})_Z = 0 \tag{5.22}$$

and, once n has been found, is a single equation for $\mathscr{F}$ with the initial condition that at $T = 0$ $\mathscr{F} = \mathscr{F}_0(Z)$. Equations (5.21) and (5.22) are most conveniently solved by the introduction of rays in terms of which (5.21) and (5.22) are reduced to a set of ordinary differential equations (Lighthill 1965). Thus n and $\mathscr{F}$ are envisaged as being propagated along the rays with the group velocity W, their values at the level Z at time T being determined in terms of their values at the initial level ζ at time $T = 0$. We have

$$\frac{dZ}{dT} = W, \quad \frac{dn}{dT} = -\left(\frac{\partial \omega'}{\partial Z} + \boldsymbol{\kappa}_H \cdot \frac{\partial \mathbf{V}}{\partial Z}\right), \tag{5.23}$$

with initial conditions $Z = \zeta$ and $n = n_0(\zeta)$. The solution of (5.23) is obtained as $Z = Z(\zeta, T)$, $n = n(\zeta, T)$ and elimination of ζ gives $n(Z, T)$. A useful relation in this context is

$$\frac{d\omega}{dT} = \boldsymbol{\kappa}_H \cdot \frac{\partial \mathbf{V}}{\partial T}. \tag{5.24}$$

$\mathscr{F}$ is found by integrating

$$d\mathscr{F}/dT = -\mathscr{F} W_Z, \tag{5.25}$$

where initially $\mathscr{F} = \mathscr{F}_0(\zeta)$; here W_Z is the partial derivative with respect to Z with T held fixed.

We shall illustrate this case by giving two examples, in both of which N^2 is a constant and $\mathbf{V} = \beta Z\mathbf{i}$, where β is a constant and $\mathbf{i}$ is a unit vector in the X direction. First, let n_0 be a constant. Then we find that

$$n = n_0 - l\beta T, \tag{5.26}$$

$$\left.\begin{aligned} \mathscr{F} &= \mathscr{F}_0(\zeta), \\ \text{where} \quad |l\beta|\,(Z-\zeta) &= \pm N\kappa_H\left\{\frac{1}{\kappa_0} - \frac{1}{(\kappa_0^2 + l^2\beta^2T^2)^{\frac{1}{2}}}\right\}. \end{aligned}\right\} \tag{5.27}$$

Here $\kappa_H = |\boldsymbol{\kappa}_H|$, κ_0 is the initial value of κ and $\pm$ is the sign of W at $T = 0$. As T increases, κ becomes infinite, $|n/\kappa|$ approaches 1, and ω' and W tend to zero; the wave energy $\mathscr{E}$, which equals $\omega'\mathscr{F}$, tends to zero with ω' and the wave packet is displaced vertically by an amount $N\kappa_H/|l\beta|\,\kappa_0$. The vertical shear has the effect of reducing the frequency and vertical scale of the waves, while their energy is transferred to the mean flow. Phillips (1969, §5.5), using a different approach and assuming a linearized theory, also obtained this solution and has discussed it in much more detail than will be given here.

Second, let the initial condition be that ω is a constant; then initially $\omega' = \omega - l\beta\zeta$ and the initial value $n_0(\zeta)$ is then determined from (5.1). The critical level is thus $Z = d$, where $d = \omega(l\beta)^{-1}$. We find that

$$\left.\begin{aligned} \omega' &= l\beta(d-Z), \\ \text{so that} \quad \frac{|n|}{\kappa} &= \left(1 - \frac{l^2\beta^2(d-Z)^2}{N^2}\right)^{\frac{1}{2}}. \end{aligned}\right\} \tag{5.28}$$

If $d_1 = N|l\beta|^{-1}$, then

$$\left.\begin{aligned} \left(\frac{d_1^2}{(d-Z)^2} - 1\right)^{\frac{1}{2}} - \left(\frac{d_1^2}{(d-\zeta)^2} - 1\right)^{\frac{1}{2}} &= \frac{\pm|l\beta|\,T}{\kappa_H}, \\ \mathscr{F}W &= \mathscr{F}_0(\zeta)\,W_0(\zeta). \end{aligned}\right\} \tag{5.29}$$

Here the choice of sign depends on whether the wave packet is propagating towards (+), or away from (−), the critical level. As T increases (assuming the sign to be plus)

$$\frac{|d-Z|}{d_1} \sim \frac{\kappa_H}{|l\beta|\,T}, \tag{5.30}$$

and the group velocity $W \sim -l^2\beta^2(d-Z)^2(N\kappa_H)^{-1}$. Thus the wave packet approaches the critical level with a decreasing velocity and will never reach it in a finite time; the wave action increases as T^2 and the wave energy as T. Thus within the wave packet the amplitude of the waves increases indefinitely. However, if initially the wave packet has a finite length, L say, then as T increases the length decreases and is proportional to LT^{-2}. The total wave action is

$$\int \mathscr{F}\,dZ = \int \mathscr{F}_0\,d\zeta, \tag{5.31}$$

where the integral on the left-hand side is over the wave packet at time T and that on the right-hand side over the wave packet at time $T = 0$. Thus the total wave action is conserved, a result which is consistent with (5.22). The total wave energy, however, is

$$\int \mathscr{E}\, dZ = \int \omega' \mathscr{F}_0(\zeta)\, d\zeta \tag{5.32}$$

and so decreases like T^{-1} as T increases. This discussion of critical-layer absorption is based on that given by Bretherton (1966) and may be compared satisfactorily with the numerical calculations of Houghton & Jones (1969).

5.2.2. *Modulations not supported by horizontal pressure gradients.* As in § 5.2.1, we suppose that ω, $\boldsymbol{\kappa}$, a and $\mathbf{V}$ are functions of Z and T alone, and so $\boldsymbol{\kappa}_H$ is again a constant. Unlike the situation examined in § 5.2.1, however, it will now be assumed that there are no horizontal pressure gradients, and so the pressure perturbation $\mathscr{P}$ is also a function of Z and T alone. The term $\nabla_H \mathscr{P}$ in (5.11) is now zero, and (5.11) implies thus that

$$\rho_0 \mathbf{V}_T + \boldsymbol{\kappa}_H \nabla . (\mathscr{F}\mathbf{c}) = 0. \tag{5.33}$$

Using (5.3) and (5.5) this simplifies to

$$\mathbf{V}_T = \mathbf{U}_T = (\boldsymbol{\kappa}_H/\rho_0)\mathscr{F}_T. \tag{5.34}$$

Thus the development of the mean flow is entirely due to the nonlinear term $\mathbf{U}$. The remaining equations are (5.1), (5.2), (5.20) and (5.22); the initial conditions at $T = 0$ are $n = n_0(Z)$, $\mathscr{F} = \mathscr{F}_0(Z)$ and $\mathbf{V} = \mathbf{V}_0(Z)$. Unlike those in § 5.2.1, these equations for n, $\mathscr{F}$ and $\mathbf{V}$ are inextricably coupled.

Although we have been unable to make any further progress when the amplitude a is $O(1)$, it is possible to develop a systematic perturbation procedure when a (and hence $\mathscr{F}$) is small. The first approximation is to replace $\mathbf{V}$ by $\mathbf{V}_0(Z)$ in (5.2) and then to solve (5.20) and (5.22); at this stage the solution procedure is the same as that given in § 5.2.1. With $\mathscr{F}$ found, (5.34) is then solved to give the second approximation for $\mathbf{V}$; the new value of $\mathbf{V}$ is then substituted into (5.2) and the process repeated. This is essentially the procedure followed by Jones & Houghton (1971) in a numerical experiment on the coupling of internal gravity waves with the mean flow. Our procedure may also be compared with the work of Lindzen & Holton (1968), whose procedure amounted to making an *ad hoc* assumption about the dependence of $\mathscr{F}$ on $\mathbf{V}$.

We shall again illustrate the situation by giving two examples, in both of which N^2 is a constant and $\mathbf{V} = \beta Z\mathbf{i}$. First, let n_0 be a constant; then the first approximation is given by (5.26) and (5.27). Equation (5.34) then implies that the second approximation for $\mathbf{V}$ is

$$\mathbf{V} = \beta Z\mathbf{i} + (\boldsymbol{\kappa}_H/\rho_0(Z))\{\mathscr{F}_0(\zeta) - \mathscr{F}_0(Z)\}, \tag{5.35}$$

where ζ is given by (5.27). As $T \to \infty$

$$|l\beta|\,|Z-\zeta| \to N\kappa_H/\kappa_0, \tag{5.36}$$

where κ_0 is the initial value of κ. Thus the ultimate effect of the nonlinear terms is to alter the vertical structure of the shear flow, described by the equation obtained by substituting (5.36) into (5.35). For example, if a_0, the initial value of a, is a constant then as $T \to \infty$

$$\mathbf{V} \to \beta Z\mathbf{i} + \tfrac{1}{2}(N^2 a_0^2/\omega_0')\{\exp(\pm N^3\kappa_H/|l\beta|\,\kappa_0) - 1\}\,\boldsymbol{\kappa}_H, \tag{5.37}$$

where ω_0' is the initial value of ω' and the $\pm$ sign is that of W at $T = 0$; in this case the nonlinear correction is simply a uniform velocity.

Second, let the initial condition be that ω is a constant; then initially $\omega' = \omega - l\beta\zeta$, and the first approximation is given by (5.28) and (5.29). Equation (5.34) then implies that the second approximation for $\mathbf{V}$ is

$$\mathbf{V} = \beta z\mathbf{i} + (\boldsymbol{\kappa}_H/\rho_0(z))\,\mathscr{F}_0(\zeta)\,(W(\zeta)/W(z)-1), \tag{5.38}$$

where ζ is given by (5.29). In this case the nonlinear correction is confined at all times to the vicinity of the wave packet, and as $T \to \infty$ the wave packet approaches the critical level $z = d$ and decreases in length as T^{-2}. The total nonlinear term produced by the wave packet is, as $T \to \infty$,

$$\frac{\boldsymbol{\kappa}_H}{\rho_0(d)}\int \mathscr{F}_0(\zeta)\,a\zeta, \tag{5.39}$$

where the integral is over the wave packet. Both these examples may be compared with the numerical experiment of Jones & Houghton (1971), in which internal gravity waves are continually generated at one level and propagate upwards to be absorbed at the critical level; the mean flow profile is continually enhanced in a band centred around the critical level.

5.3. *Two-dimensional modulations*

We now suppose that ω, $\boldsymbol{\kappa}$, a and $\mathbf{V}$ are functions of X, Z and T only, and also that $\mathbf{V} = V\mathbf{i}$ and $\boldsymbol{\kappa}_H = l\mathbf{i}$; thus the entire motion is confined to planes perpendicular to the Y axis. Equation (5.3) implies that $V = V(Z,T)$ and equation (5.5) (or equation (5.14)) imposes no further restriction on V. However (5.11) becomes

$$\rho_0 V_T + \nabla.(\mathscr{F}l\mathbf{c}) = -\mathscr{P}_X \tag{5.40}$$

and, if it is assumed that the wave packet is confined to a bounded region, $\rho_0 V_T = -\mathscr{P}_X$ in the region outside the wave packet. Thus either the wave packet is supported by a horizontal pressure gradient or $V_T = 0$ in the region outside the wave packet. The case when $V \equiv 0$ everywhere has been discussed by Bretherton (1969).

5.4. *Small amplitude modulations*

In the absence of any special geometrical symmetries, we shall consider only the case of small perturbations to the quasi-steady modulations (described in § 5.1). We let the subscript zero denote the quasi-steady modulation described by the phase Θ_0, stream function Ψ_0 and wave action $\mathscr{F}_0$, where

$$\omega_0 = -\Theta_{0T}, \quad \boldsymbol{\kappa}_0 = \nabla\Theta_0, \quad \mathbf{V}_0 = \nabla\times(\Psi_0\mathbf{k}) \tag{5.41}$$

are, like $\mathscr{F}_0$, functions of Z only. The perturbations $\hat{\Theta}$, $\hat{\Psi}$ and $\mathscr{F}_0$ are defined by

$$\Theta = \Theta_0 + \hat{\Theta}, \quad \Psi = \Psi_0 + \hat{\Psi}, \quad \mathscr{F} = \mathscr{F}_0 + \hat{\mathscr{F}}. \tag{5.42}$$

C and D are already perturbed quantities, as their counterparts in the quasi-steady solution, C_0 and D_0, are identically zero. When (5.42) is substituted into

(5.1), (5.3) and (5.16), and the subsequent equations for $\hat{\Theta}$, $\hat{\Psi}$ and $\hat{\mathscr{F}}$ are linearized, we find that

$$\left.\begin{aligned} 0 &= \hat{\Theta}_T + (\mathbf{c}_0 + \mathbf{V}_0) . \nabla\hat{\Theta} + \mathbf{k} \times \boldsymbol{\kappa}_0 . \nabla\hat{\Psi}, \\ 0 &= \hat{\mathscr{F}}_T + (\mathbf{c}_0 + \mathbf{V}_0) . \nabla\hat{\mathscr{F}} + \mathscr{F}_0 \nabla . \hat{\mathbf{c}} + \mathscr{F}_{0Z}(\hat{\mathbf{c}} . \mathbf{k} - W_0 \hat{\mathscr{F}}/\mathscr{F}_0), \\ 0 &= \nabla_H^2 \hat{\Psi} + (1/\rho_0) \nabla_H \hat{\mathscr{F}} . \boldsymbol{\kappa}_0 \times \mathbf{k}. \end{aligned}\right\} \tag{5.43}$$

Here $\mathbf{c}_0$ is the group velocity for the quasi-steady modulation, viz. $\nabla_{\kappa_0}\omega_0'$, and is given by (3.19); $W_0 = \mathbf{c}_0 . \mathbf{k}$; and $\hat{\mathbf{c}}$ is the perturbed group velocity and is given by

$$\hat{\mathbf{c}} = (\hat{\boldsymbol{\kappa}} . \nabla_{\kappa_0}) \mathbf{c}_0. \tag{5.44}$$

The set (5.43) is a set of linear equations for $\hat{\Theta}$, $\hat{\Psi}$ and $\hat{\mathscr{F}}$. However, they are inhomogeneous equations as the coefficients $\mathscr{F}_0$ etc. are functions of Z. We are thus prevented from seeking exponential solutions and hence determining the stability of the quasi-steady solution. We adopt, therefore, the criterion proposed by Whitham (1967), in the context of water waves, and associate instability with the existence of imaginary characteristics for (5.43). However, instead of directly seeking the characteristics of (5.43), we shall adopt the equivalent procedure of seeking an asymptotic solution to (5.43) which is rapidly varying with respect to the $\mathbf{X}$, T co-ordinates, although still slowly varying with respect to the $\mathbf{x}$, t co-ordinates. Thus we put

$$\left.\begin{aligned} \hat{\Theta} &\sim \mathrm{Re}\left[\mu\hat{\Theta}_1 \exp\left(\frac{iS(\mathbf{X}, T)}{\mu}\right)\right], \\ \hat{\Psi} &\sim \mathrm{Re}\left[\mu\hat{\Psi}_1 \exp\left(\frac{iS(\mathbf{X}, T)}{\mu}\right)\right], \\ \hat{\mathscr{F}} &\sim \mathrm{Re}\left[\hat{\mathscr{F}}_1 \exp\left(\frac{iS(\mathbf{X}, T)}{\mu}\right)\right], \end{aligned}\right\} \tag{5.45}$$

where μ is a small parameter such that ϵ is $o(\mu)$ and the asymptotic solution holds as $\mu \to 0$. The local frequency σ and the local wavenumber vector $\boldsymbol{\nu}$ of this asymptotic solution are given by

$$\sigma = -S_T, \quad \boldsymbol{\nu} = \nabla S. \tag{5.46}$$

Equation (5.45) has been constructed such that the derivatives of $\hat{\Theta}$ and $\hat{\Psi}$ are $O(1)$ as $\mu \to 0$, while the derivatives of $\hat{\omega}$, $\hat{\boldsymbol{\kappa}}$, $\hat{\mathbf{V}}$ (i.e. $\nabla \times (\hat{\Psi}\mathbf{k})$) and $\hat{\mathscr{F}}$ are $O(\mu^{-1})$. When (5.45) is substituted into (5.43) and the terms of $O(\mu^{-1})$ equated to zero, we find that

$$\left.\begin{aligned} 0 &= \{-\sigma + (\mathbf{c}_0 + \mathbf{V}_0) . \boldsymbol{\nu}\} \hat{\Theta}_1 + (\boldsymbol{\nu} . \mathbf{k} \times \boldsymbol{\kappa}_0) \hat{\Psi}_1, \\ 0 &= -\mathscr{F}_0 \nu_i \nu_j \frac{\partial^2 \omega_0'}{\partial \kappa_{0i} \partial \kappa_{0j}} \hat{\Theta}_1 + \{-\sigma + (\mathbf{c}_0 + \mathbf{v}_0) . \boldsymbol{\nu}\} i\hat{\mathscr{F}}_1, \\ 0 &= -\nu_H^2 \hat{\Psi}_1 + (\boldsymbol{\nu} . \mathbf{k} \times \boldsymbol{\kappa}_0)\, i\hat{\mathscr{F}}_1/\rho_0. \end{aligned}\right\} \tag{5.47}$$

Here the ν_i, $i = 1, 2, 3$, are the components of $\boldsymbol{\nu}$, the κ_{0i}, $i = 1, 2, 3$, are the components of $\boldsymbol{\kappa}_0$ and $\nu_H^2 = \nu_1^2 + \nu_2^2$. For a non-trivial solution the determinant of (5.47) must vanish, and so

$$\{-\sigma + (\mathbf{c}_0 + \mathbf{V}_0) . \boldsymbol{\nu}\}^2 = -\frac{\mathscr{F}_0}{\rho_0} \frac{(\boldsymbol{\nu} . \mathbf{k} \times \boldsymbol{\kappa}_0)^2}{\nu_H^2} \nu_i \nu_j \frac{\partial^2 \omega_0'}{\partial \kappa_{0i} \partial \kappa_{0j}}. \tag{5.48}$$

For instability this equation will have complex solutions for σ for real $\mathbf{\nu}$, so the criterion for instability is that the right-hand side of (5.48) be positive, i.e

$$\left.\begin{aligned} \mathbf{\nu}.\mathbf{k}\times\mathbf{\kappa}_0 \neq 0, \\ \omega_0'\nu_i\nu_j\frac{\partial^2\omega_0'}{\partial\kappa_{0i}\partial\kappa_{0j}} > 0. \end{aligned}\right\} \tag{5.49}$$

The first condition requires $\mathbf{\nu}$ to be perpendicular to both $\mathbf{k}$ and $\mathbf{\kappa}_0$, while the second condition implies that

$$\nu_\perp^2 + \frac{(\mathbf{\nu}.\mathbf{\kappa})^2}{\kappa^2} - \left\{\frac{(\mathbf{\nu}.\mathbf{\kappa})^2}{\kappa^2} + \frac{(\mathbf{\nu}.\mathbf{c})\,\omega'}{|\mathbf{c}|^2\kappa}\right\}^2 > 0, \tag{5.50}$$

where $\nu_\perp$ is the component of $\mathbf{\nu}$ perpendicular to the plane of $\mathbf{k}$ and $\mathbf{\kappa}_0$. Thus there will be instability for large enough values of $\nu_\perp$; when $\mathbf{\nu}.\mathbf{c} = 0$, there will be instability for any non-zero value of $\nu_\perp$. Finally, it may be observed that the right-hand side of (5.48) is proportional to $\mathscr{F}_0$ and would vanish for a linearized internal gravity wave; the instability criterion is therefore due to the nonlinear aspects of the wave.

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